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Specific Properties of a Subclass of Univalent Functions with Finite Fixed Coefficients

Dr. R. Arumugam
Jeppiaar Engineering College, TN. India

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions f defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. Such a function has the Taylor series expansion about the origin as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \Delta. \tag{1.1}$$

We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions that are univalent. Goodman [2, 3] defined and studied the subclass of uniformly starlike and uniformly convex functions. Murugusundaramoorthy et al. [4] extended the study of the above subclass by fixing the second coefficient. In recent times, researchers [1, 5] have defined new subclasses of $\mathcal{S}$ by fixing a finite number of coefficients of functions. In this paper, we consider the subclass $\mathcal{SD}(\alpha)$ of $\mathcal{S}$ by fixing finitely many coefficients and properties of the functions in this subclass are examined.

$\mathcal{T}$ denotes the subclass of $\mathcal{S}$ consisting of functions with negative coefficients. Thus if $f \in \mathcal{T}$ then

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.2}$$

Definition 1.1 ([6]). A function $f \in \mathcal{S}$ is in the class $\mathcal{SD}(\alpha)$ if it satisfies the analytic criteria

$$Re\left\{\frac{f(z)}{z}\right\} \geq \alpha\left|f'(z) - \frac{f(z)}{z}\right|, \alpha \geq 0. \tag{1.3}$$

The intersection of the classes $\mathcal{T}$ and $\mathcal{SD}(\alpha)$ is denoted by $\mathcal{TSD}(\alpha)$. We now state a necessary and sufficient condition for the functions in $\mathcal{S}$ to be in $\mathcal{TSD}(\alpha)$.

Theorem 1.2. *A function of the form* (1.2) *is in the class* $\mathcal{TSD}(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} [1 + \alpha(n-1)]|a_n| \leq 1, \ \alpha \geq 0. \tag{1.4}$$

Theorem 1.2. *A function of the form (1.2) is in the class $\mathcal{TSD}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]|a_n| \leq 1,\ \alpha \geq 0. \tag{1.4}$$

Proof. Assume that f of the form (1.2) satisfies (1.4). Then

$Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$

$\geq 1 - \left|\frac{f(z)}{z} - 1\right| - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$

$= 1 - \sum_{n=2}^{\infty}|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n| \geq 0$. Hence $f \in \mathcal{TSD}(\alpha)$.

Conversely,

$Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right| > 0.$

which implies $Re\{1 - \sum_{n=2}^{\infty}|a_n|z^{n-1}\} - \alpha|\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}| > 0$

Letting z to take real values and as $|z| \to 1$, we get

$1 - \sum_{n=2}^{\infty}|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n| \geq 0.$

which implies $\sum_{n=2}^{\infty}[1+\alpha(n-1)]|a_n| \leq 1.$ □

Corollary 1.3. *For $f \in \mathcal{TSD}(\alpha)$*

$$a_n \leq \frac{1}{1+\alpha(n-1)}, n \geq 2. \tag{1.5}$$

We now introduce the subclass $\mathcal{TSD}(\alpha, p_k)$ of $\mathcal{TSD}(\alpha)$. This class consists of all those functions in $\mathcal{TSD}(\alpha)$ which are of the form

$$f(z) = z - \sum_{i=2}^{k}\frac{p_i}{1+\alpha(i-1)}z^i - \sum_{n=k+1}^{\infty}a_n z^n. \tag{1.6}$$

Several interesting properties of the functions in this class are proved in the subsequent sections.

2. Coefficient Estimates

We now prove the coefficient estimate for functions in the class $\mathcal{TSD}(\alpha, p_k)$.

Theorem 2.1. *A function of the form (1.6) is in the class $\mathcal{TSD}(\alpha, p_k)$ if and only if*

$$\sum_{n=k+1}^{\infty}[1+\alpha(n-1)]a_n \leq 1 - \sum_{i=2}^{k}p_i, \tag{2.1}$$

where $\alpha \geq 0,\ 0 \leq p_i \leq 1$ and $0 \leq \sum_{i=2}^{k}p_i \leq 1$. The result is sharp.

Proof. By (1.5),

$$a_i = \frac{p_i}{1+\alpha(i-1)},\ i = 2,3,...,k,\ 0 \leq p_i \leq 1,\ 0 \leq \sum_{i=2}^{k}p_i \leq 1. \tag{2.2}$$

which implies $\sum_{i=2}^{k}p_i + \sum_{n=k+1}^{\infty}[1+\alpha(n-1)]a_n \leq 1.$

Conversely,

$Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$

$\geq 1 - \left|\frac{f(z)}{z} - 1\right| - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$

$= 1 - \sum_{n=2}^{\infty} |a_n| - \alpha \sum_{n=2}^{\infty}(n-1)|a_n|$

$= 1 - \sum_{i=2}^{k}[1 + \alpha(i-1)]|a_i| - \sum_{n=k+1}^{\infty}[1 + \alpha(n-1)]|a_n|$

$= 1 - \sum_{i=2}^{k} p_i - \sum_{n=k+1}^{\infty}[1 + \alpha(n-1)]|a_n|$, by (2.2)

≥ 0, by (2.1).

Thus $f \in \mathcal{TSD}(\alpha, p_k)$. The sharpness of the result follows by taking

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{1 + \alpha(i-1)} z^i - \frac{\left(1 - \sum_{i=2}^{k} p_i\right)}{1 + \alpha(n-1)} z^n, \; n \geq 1. \tag{2.3}$$

□

The following corollary is a consequence of Theorem 2.1.

Corollary 2.2. *If f is in the class $\mathcal{TSD}(\alpha, p_k)$, then*

$$a_n \leq \frac{1 - \sum_{i=2}^{k} p_i}{1 + \alpha(n-1)}, \; n \geq k+1. \tag{2.4}$$

The result is sharp for the functions f given by (2.3).

Theorem 3.1. *The class $\mathcal{TSD}(\alpha, p_k)$ is convex.*

Proof. Let f, g be two functions in $\mathcal{TSD}(\alpha, p_k)$. Then

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{1 + \alpha(n-1)} z^i - \sum_{n=k+1}^{\infty} a_n z^n,$$

$$g(z) = z - \sum_{i=2}^{k} \frac{p_i}{1 + \alpha(n-1)} z^i - \sum_{n=k+1}^{\infty} b_n z^n,$$

where $0 \leq p_i \leq 1, 0 \leq \sum_{i=2}^{k} p_i \leq 1$.

Define $h(z) = \lambda f(z) + (1-\lambda)g(z)$. Then $h(z) = z - \sum_{i=2}^{\infty} \frac{p_i}{1+\alpha(i-1)} z^i - \sum_{n=k+1}^{\infty}[\lambda a_n + (1-\lambda)b_n]z^n$.

Now,

$\sum_{n=k+1}^{\infty}[1 + \alpha(n-1)][\lambda a_n + (1-\lambda)b_n]$

$= \lambda \sum_{n=k+1}^{\infty}[1 + \alpha(n-1)]a_n + (1-\lambda)\sum_{n=k+1}^{\infty}[1 + \alpha(n-1)]b_n$

$\leq \lambda(1 - \sum_{i=2}^{k} p_i) + (1-\lambda)(1 - \sum_{i=2}^{k} p_i)$

$= 1 - \sum_{i=2}^{k} p_i$

which implies $h(z) \in \mathcal{TSD}(\alpha, p_k)$. □

Theorem 3.2. *Let*

$$f_k(z) = z - \sum_{i=2}^{k} \frac{p_i}{1+\alpha(n-1)} z^i \tag{3.1}$$

and

$$f_n(z) = z - \sum_{i=2}^{k} \frac{p_i}{[1+\alpha(i-1)} z^i - \frac{\left(1 - \sum_{i=2}^{k} p_i\right)}{1+\alpha(n-1)} z^n, n \geq k+1. \tag{3.2}$$

Then $f \in \mathcal{TSD}(\alpha, p_k)$ *if and only if* f *can be expressed in the form*

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z), \tag{3.3}$$

where $\lambda_n \geq 0,\ (n \geq k)$ *and* $\sum_{n=k}^{\infty} \lambda_n = 1$.

Proof. Suppose $f \in \mathcal{T}$ can be expressed in the form (3.3). Then

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{1+\alpha(i-1)} z^i - \sum_{n=k+1}^{\infty} \frac{\lambda_n[1 - \sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} z^n. \tag{3.4}$$

Now, $\sum_{n=k+1}^{\infty} [1+\alpha(n-1)] \frac{\lambda_n[1-\sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} = \sum_{n=k+1}^{\infty} \lambda_n [1 - \sum_{i=2}^{k} p_i]$

$= [1 - \sum_{i=2}^{k} p_i] \sum_{n=k+1}^{\infty} \lambda_n$

$= [1 - \sum_{i=2}^{k} p_i](1 - \lambda_k)$

$\leq 1 - \sum_{i=2}^{k} p_i$

which implies $f \in \mathcal{TSD}(\alpha, p_k)$.

Conversely, for $n \geq k+1$, set

$$\lambda_n = \frac{[1+\alpha(n-1)]a_n}{1 - \sum_{i=2}^{\infty} p_i}, n \geq k+1. \tag{3.5}$$

and

$$\lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n. \tag{3.6}$$

Then f can be represented as $f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z)$. □

Corollary 3.3. *The extreme points of the class $\mathcal{TSD}(\alpha, p_k)$ are the functions $f_n, (n \geq k)$ given by (3.1) and (3.2).*

4. Integral Operator

The Alexander Operator for the functions in the class $\mathcal{S}$ is defined as

$$\mathcal{I}(f) = \int_0^z \frac{f(t)}{t} dt. \tag{4.1}$$

This operator maps the class of starlike functions onto the class of convex functions. The effect of this operator on the functions in the class $\mathcal{TSD}(\alpha, p_k)$ is given in the following theorem.

Theorem 4.1. *Let f defined by (1.5) be in the class $\mathcal{TSD}(\alpha, p_k)$. Then $\mathcal{I}(f)$ belongs to the class $\mathcal{TSD}(\alpha, q_k)$ where $q_k = \frac{p_k}{k}$.*

Proof. We have

$$\mathcal{I}(f) = z - \sum_{i=2}^{k} \frac{q_i}{1+\alpha(i-1)} z^i - \sum_{n=k+1}^{\infty} \frac{a_n}{n} z^n. \tag{4.2}$$

Now,
$\sum_{n=k+1}^{\infty} [1+\alpha(n-1)] \frac{a_n}{n}$

$\leq \frac{1}{k+1} \sum_{n=k+1}^{\infty} [1+\alpha(n-1)] a_n$

$\leq \frac{1}{k+1} (1 - \sum_{i=2}^{k} p_i)$

$= \frac{1}{k+1} - \sum_{i=2}^{k} \frac{p_i}{k+1}$

$< 1 - \sum_{i=2}^{k} \frac{p_i}{i}$

which implies $\mathcal{I}(f) \in \mathcal{TSD}(\alpha, q_k)$. □

5. Radius of Starlikeness and Convexity

In this section, we derive the radii results for the function in the class $\mathcal{TSD}(\alpha, p_k)$ to be starlike or convex of order β.

Theorem 5.1. *The function given by (1.5) in the class $\mathcal{TSD}(\alpha, p_k)$ is starlike of order β $(0 \leq \beta \leq 1)$ in the disk $|z| < r_1$ where r_1 is the largest value which satisfies*

$$\sum_{i=2}^{\infty} \left[\frac{(2-i)-\beta)}{1+\alpha(i-1)}\right] p_i r^{i-1} + \frac{((2-n)-\beta)[1-\sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} r^{n-1} \leq \beta. \tag{5.1}$$

Proof. $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{i=2}^{k}[1-i]\frac{p_i}{1+\alpha(i-1)} r^{i-1} - \sum_{n=k+1}^{\infty}(n-1)a_n r^{n-1}}{1-\sum_{i=2}^{k} \frac{p_i}{1+\alpha(i-1)} r^{i-1} - \sum_{n=k+1}^{\infty} a_n r^{n-1}}$

which is less than or equal to $1-\beta$ for $|z| \leq r$ if and only if

$$\sum_{i=2}^{k} \frac{(2-i)-\beta}{1+\alpha(i-1)} p_i r^{i-1} + \sum_{n=k+1}^{\infty} ((2-n)-\beta) a_n r^{n-1} \leq 1-\beta. \tag{5.2}$$

By Corollary 2.2, we may set

$$a_n = \frac{[1 - \sum_{i=2}^{k} p_i]}{1 + \alpha(n-1)}\lambda_n, \;\; n \geq k+1, \tag{5.3}$$

where $\lambda_n \geq 0 \;\; (n \geq k+1), \; \sum_{n=k+1}^{\infty} \lambda_n \leq 1.$
For each fixed r, choosing an integer $n_0 = n_0(r)$ for which $\frac{((2-n)-\beta)r^{n-1}}{1+\alpha(n-1)}$ is a maximum, we obtain

$$\sum_{n=k+1}^{\infty} (n-\beta)a_n r^{n-1} \leq \frac{((2-n_0)-\beta)[1 - \sum_{i=2}^{k} p_i]}{1+\alpha(n_0-1)} r^{n_0-1}. \tag{5.4}$$

Hence f is starlike of order β in $|z| \leq r_1$ provided

$$\sum_{i=2}^{k} \frac{(2-i)-\beta}{1+\alpha(i-1)} p_i r^{i-1} + \frac{((2-n_0)-\beta)[1 - \sum_{i=2}^{k} p_i]}{1+\alpha(n_0-1)} r^{n_0-1} \leq 1-\beta. \tag{5.5}$$

We find the value of r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{i=2}^{k} \frac{(2-i)-\beta}{1+\alpha(i-1)} p_i r_0^{i-1} + \frac{(2-n_0)-\beta)[1 - \sum_{i=2}^{k} p_i]}{1+\alpha(n_0-1)} r_0^{n_0-1} = 1-\beta \tag{5.6}$$

which is the radius of starlikeness of order β for functions in the class $\mathcal{TSD}(\alpha, p_k)$. □

In the following theorem we obtain the radius of convexity for functions in the class $\mathcal{TSD}(\alpha, p_k)$.

Theorem 5.2. *The function given by (1.5) in the class $\mathcal{TSD}(\alpha, p_k)$ is convex of order β $(0 \leq \beta \leq 1)$ in the disk $|z| < r_2$ where r_2 is the largest value which satisfies*

$$\sum_{i=2}^{\infty} \left[\frac{i(i-\beta)}{1+\alpha(i-1)}\right] p_i r^{i-1} + \frac{n(n-\beta)[1 - \sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} r^{n-1} \leq \beta. \tag{5.7}$$

Proof. $\left|\frac{zf''(z)}{f(z)}\right| \leq \frac{\sum_{i=2}^{k} \frac{i(i-1)p_i}{1+\alpha(i-1)} r^{i-1} + \sum_{n=k+1}^{\infty} n(n-1)a_n r^{n-1}}{1 - \sum_{i=2} k \frac{ip_i}{1+\alpha(i-1)} - \sum_{n=k+1}^{\infty} na_n r^{n-1}}$
which is less than or equal to $1-\beta$ for $|z| < r$ if and only if

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)}{1+\alpha(i-1)} r^{i-1} + \sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \leq 1-\beta. \tag{5.8}$$

Using Corollary 2.2 and for each fixed r, choosing an integer $n_0 = n_0(r)$ for which $\frac{n_0(n_0-\beta)r^{n-1}}{1+\alpha(n_0-1)}$ is a maximum, we get

$$\sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \leq \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{1+\alpha(n_0-1)} r_0^{n-1}. \tag{5.9}$$

Hence f is convex of order β in $|z| < r_2$ provided

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)}{1+\alpha(i-1)} r^{i-1} + \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{1+\alpha(n_0-1)} r^{n-1} \leq 1-\beta. \tag{5.10}$$

We find the value of r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)}{1+\alpha(i-1)} r_0^{i-1} + \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{1+\alpha(n_0-1)} r_0^{n-1} \leq 1-\beta \tag{5.11}$$

which is the radius of convexity of order β for functions in the class $\mathcal{TSD}(\alpha, p_k)$. □

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