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Convergence In N – Inner Product Space

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Abstract: The concept of convergence in an nnn-inner product space extends the classical understanding of convergence in vector spaces. This study explores the foundational principles underlying nnn-inner product spaces and their associated nnn-normed spaces. It delves into the properties and structures that distinguish nnn-inner product spaces from traditional inner product spaces, emphasizing the unique convergence criteria that arise in these higher-dimensional frameworks. By examining the interplay between nnn-inner products and nnn-norms, the paper contributes to a deeper understanding of the geometric and analytic characteristics of nnn-inner product spaces, paving the way for further research in functional analysis and its applications.

Keywords: nnn-inner product, nnn-inner product space, nnn-normed product space.

1.1 Introduction

The study of convergence in various mathematical structures is a cornerstone of functional analysis, providing essential insights into the behavior of sequences and their limits within these spaces. In particular, nnn-inner product spaces, an extension of the classical inner product spaces, present a rich framework for examining convergence properties in higher-dimensional contexts. These spaces generalize the concept of an inner product by considering multiple vectors simultaneously, which leads to the development of nnn-norms and corresponding notions of convergence.

In this paper, we focus on two critical types of convergence in nnn-inner product spaces: strong convergence and weak convergence. These concepts, well-established in the context of standard

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inner product spaces, acquire new dimensions of complexity and intrigue when extended to nnn-inner product spaces. Understanding the relationship between these types of convergence is not only mathematically significant but also essential for the broader application of nnn-inner product spaces in areas such as functional analysis, quantum mechanics, and other fields where higher-order interactions are modeled.

We begin by defining strong and weak convergence in the context of nnn-inner product spaces. Strong convergence, in this setting, refers to the convergence of a sequence in terms of the nnn-norm induced by the nnn-inner product, while weak convergence pertains to the convergence of sequences when evaluated under the nnn-inner product itself. These definitions extend the familiar notions from inner product spaces but incorporate the complexities of nnn-dimensional interactions.

One of the central results we establish in this paper is that strong convergence in nnn-inner product spaces necessarily implies weak convergence. This result aligns with the classical theory in inner product spaces, where strong convergence implies weak convergence due to the continuity of the inner product with respect to the norm. However, the reverse implication—whether weak convergence implies strong convergence—does not generally hold. To demonstrate this, we provide a counterexample that clearly illustrates the distinction between strong and weak convergence in nnn-inner product spaces.

To further support our analysis, we invoke an analogue of Parseval's identity, traditionally used in inner product spaces to relate the norm of a vector to the sum of the squares of its coefficients in an orthonormal basis. In the context of nnn-inner product spaces, this identity is extended to reflect the structure of these spaces, providing a powerful tool for understanding the interplay between strong and weak convergence.

Through this exploration, we aim to deepen the understanding of convergence in nnn-inner product spaces and clarify the relationships between these different notions of convergence. Our results contribute to the broader mathematical theory of nnn-inner product spaces and open avenues for further research into their applications and underlying principles.

Definitions and Notes

1.2 Definition of n-Inner Product Space

Let n be a positive integer and X be a vector space of dimension $d \geq n$ (where d can be infinite) over the field of real numbers R. An n-inner product is a real-valued function $\langle \cdot, \cdot \mid \cdot, ..., \cdot \rangle$ defined on the Cartesian product $X \times X \times ... \times X = X^{(n+1)}$ that satisfies the following conditions:

(I1) Positivity and Linear Independence:
$\langle x_1, x_1 \mid x_2, ..., x_n\rangle \geq 0$ for any $x_1, x_2, ..., x_n \in X$,
and
$\langle x_1, x_1 \mid x_2, ..., x_n\rangle = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent vectors.

This condition ensures that the n-inner product is non-negative and reflects the linear independence of the vectors involved.

(I2) Symmetry under Permutation:
$\langle x\_\{i_1\}, x\_\{i_1\} \mid x\_\{i_2\}, ..., x\_\{in\}\rangle = \langle x_1, x_1 \mid x_2, ..., x_n\rangle$ for every permutation $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$.

This symmetry condition ensures that the n-inner product remains invariant under any permutation of its arguments.

(I3) Symmetry of the First Two Arguments:
$\langle x, y \mid x_2, ..., x_n\rangle = \langle y, x \mid x_2, ..., x_n\rangle$ for all $x, y, x_2, ..., x_n \in X$.

This property ensures that the n-inner product is symmetric in its first two arguments.

(I4) Linearity in the First Argument:
$\langle \alpha x, y \mid x_2, ..., x_n\rangle = \alpha\langle x, y \mid x_2, ..., x_n\rangle$ for all $x_2, ..., x_n \in X$ and $\alpha \in R$.

This linearity condition ensures that the n-inner product is linear with respect to scalar multiplication in its first argument.

(I5) Additivity in the First Argument:
$\langle x + y, z \mid x_2, ..., x_n\rangle = \langle x, z \mid x_2, ..., x_n\rangle + \langle y, z \mid x_2, ..., x_n\rangle$ for all $x, y, z, x_2, ..., x_n \in X$.

This additivity property ensures that the n-inner product is additive in its first argument.

A function $\langle\cdot, \cdot \mid \cdot, ..., \cdot\rangle$ that satisfies these conditions is called an n-inner product on X, and the corresponding pair $(X, \langle\cdot, \cdot \mid \cdot, ..., \cdot\rangle)$ is referred to as an n-inner product space.

1.3 Strong Convergence in n-Inner Product Space

Let $(X, \langle\cdot, \cdot \mid \cdot, ..., \cdot\rangle)$ be an n-inner product space, and let $\|\cdot,...,\cdot\|$ be the induced n-norm. A sequence (x_k) in X is said to converge strongly to a point $x \in X$ if
$\|x_k - x, x_2, ..., x_n\| \to 0$ for every $x_2, ..., x_n \in X$.

In such a case, we write $x_k \to x$. This definition extends the classical notion of strong convergence in normed spaces to the context of n-inner product spaces, where convergence is determined by the n-norm induced by the n-inner product.

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1.4 Weak Convergence in n-Inner Product Space

A sequence (x_k) in X is said to converge weakly to x ∈ X if $\langle x_k - x, y \mid x_2, ..., x_n\rangle \to 0$ for every $y, x_2, ..., x_n \in X$. This definition captures the idea of weak convergence in n-inner product spaces, where convergence is determined by the behavior of the sequence under the n-inner product, rather than the n-norm.

Note 1.3: Strong and Weak Convergence of Linear Combinations

If the sequences (x_k) and (y_k) converge strongly or weakly to x and y respectively, then for any α, β ∈ R, the sequence $(\alpha x_k + \beta y_k)$ converges strongly or weakly to $\alpha x + \beta y$. This note emphasizes that both strong and weak convergence are preserved under linear combinations of convergent sequences.

Note 1.5: Continuity of the n-Norm

If $x_k \to x$ strongly, then
$\|x_k, x_2, ..., x_n\| \to \|x, x_2, ..., x_n\|$ for every $x_2, ..., x_n \in X$.
This indicates that the n-norm $\|\cdot,...,\cdot\|$ is continuous in the first variable. By property (N2) of n-norms, this continuity extends to each variable, highlighting the smoothness of the n-norm in n-inner product spaces.

Note: 1.6

If $x_k \to x$ and $y_k \to y$, then by the triangle inequality for real numbers and the Cauchy – Schwarz inequality for the n – inner product we have $|\langle x_k, y_k|x_2, \dots, x_n\rangle - \langle x, y|x_2, \dots, x_n\rangle| \leq |\langle x_k - x, y|x_2, \dots, x_n\rangle|$

$$+|\langle x_k - x, y_k - y|x_2, \dots, x_n\rangle| + |\langle x, y_k - y|x_2, \dots, x_n\rangle|$$

$$\leq \|x_k - x, x_2, \dots, x_n\| \cdot \|y, x_2, \dots, x_n\|$$

$$+\|x_k - x, x_2, \dots, x_n\| \cdot \|y_k - y, x_2, \dots, x_n\|$$

$$+\|x, x_2, \dots, x_n\| \cdot \|y_k - y, x_2, \dots, x_n\|$$

Where $\langle x_k, y_k|x_2, \dots, x_n\rangle \to \langle x, y|x_2, \dots, x_n\rangle$. This shows that $\langle \cdot,\cdot \mid \cdot, \dots, \cdot\rangle$ is continuous in the first two variables.

Now we come to our main results. The first theorem below tells us that a sequence cannot converge weakly to two distinct points.

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Theorem: 1.7

If (x_k) converges weakly to x and x' simultaneously, then $x = x'$.

Proof:

By hypothesis and property (15) of n – inner products, we have

$\langle x_k, y|x_2, \dots, x_n\rangle \to \langle x, y|x_2, \dots, x_n\rangle$ and at the same time

$\langle x_k, y|x_2, \dots, x_n\rangle \to \langle x', y|x_2, \dots, x_n\rangle$ for every $y, x_2, \dots, x_n \in X$. By the uniqueness of the limit of a sequence of real numbers, we must have $\langle x, y|x_2, \dots, x_n\rangle = \langle x', y|x_2, \dots, x_n\rangle$

or $\langle x - x', y|x_2, \dots, x_n\rangle = 0$

for every $y, x_2, \dots, x_n \in X$. In particular, by taking $y = x - x'$ we obtain $\|x - x', x_2, \dots, x_n\| = 0$

for every $y, x_2, \dots, x_n \in X$.

By property (N1) of 2 – norms and elementary linear algebra, this can only happens if $x - x' = 0$ or $x = x'$.

The next proposition says that the strong convergence implies the weak convergence.

Theorem: 1.8

If (x_k) converges strongly to x then it converges weakly to x.

Proof:

By the Cauchy – Schwarz inequality, we have

$|\langle x_k - x, y|x_2, \dots, x_n\rangle| \le \|x_k - x, x_2, \dots, x_n\| \cdot \|y, x_2, \dots, x_n\|$

for every $y, x_2, \dots, x_n \in X$. Since by hypothesis the right – hand side tends to 0 for every $y, x_2, \dots, x_n \in X$, so does the left – hand side.

Corollary: 1.9

A sequence cannot converge strongly to two distinct points.

The terminology that we use suggests that there are sequences that converge weakly but do not converge strongly. Here is one example that invokes an analogue of Parseval's identity.

Example: 1.10

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Let $(X, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space of infinite dimension and (e_k), indexed by N, be an orthonormal basis for X. Then, for each x and $z \in X$, we have

$$\sum_k \langle x, e_k\rangle \langle z, e_k\rangle = \langle x, z\rangle$$

In particular, if $x = z$, then we have Parseval's identity

$\sum_k \langle x, e_k\rangle^2 = \|x\|^2$ where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ denotes the induced form.

Now equip X with the standard n – inner product $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle$ as given previously in the introduction. Then, for each $x, z_2, \ldots, z_n \in X$, we have the following analogue of Parseval's identity

$$\sum_k \langle x, e_k|z_2, \ldots, z_n\rangle^2 = \|x, z_2, \ldots, z_n\|^2 \| z_2, \ldots, z_n\|^2{}_{n-1}$$

Where $\|\cdot,\ldots,\cdot\|_{n-1}$ denotes the standard $(n-1)$ – norm on X. For $n = 2$, the identity can be verified easily as follows

$$\begin{aligned}
\sum_k \langle x, e_k|z\rangle^2 &= \sum_k [\langle x, e_k\rangle\|z\|^2 - \langle x, z\rangle\langle z, e_k\rangle]^2 \\
&= \sum_k [\langle x, e_k\rangle^2\|z\|^4 - 2\langle x, e_k\rangle\langle z, e_k\rangle\langle x, z\rangle\|z\|^2 + \langle x, z\rangle^2\langle z, e_k\rangle^2] \\
&= \|x\|^2\|z\|^4 - 2\langle x, z\rangle^2\|z\|^2 + \langle x, z\rangle^2\|z\|^2 \\
&= [\|x\|^2\|z\|^2 - \langle x, z\rangle^2] \cdot \|z\|^2 \\
&= \|x, z\|^2\|z\|^2
\end{aligned}$$

Because of Parseval's identity, we must have $\langle x, e_k|z_2, \ldots, z_n\rangle \to 0$ for every $x, z_2, \ldots, z_n \in X$, that is, (e_k) converges weakly to 0. Now, for each $k \in N$ and $z_2, \ldots, z_n \in X$, denote by $e_k{}^*$ the orthogonal projection of e_k on the subspace spanned by $z_2, \ldots, z_n$. Then $\|e_k - e_k{}^*\| \to 1$, where

$$\|e_k, z_2, \ldots, z_n\| = \|e_k - e_k{}^*\| \cdot \| z_2, \ldots, z_n\|_{n-1} \to \| z_2, \ldots, z_n\|_{n-1} \neq 0$$

Whenever $z_2, \ldots, z_n$ are linearly independent. This shows that (e_k) does not converge strongly to 0 in X.

Special Cases

On any n – inner product space $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$, we can define an inner product $\langle\cdot,\cdot\rangle$ with respect to a linearly independent set $\{a_1, \ldots, a_n\} \subseteq X$ by

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$$\langle x, y\rangle := \sum_{\{i_2,\ldots,i_n\}\subseteq\{1,\ldots,n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n}\rangle$$

And put $\|\cdot\| = \langle\cdot\,,\cdot\rangle^{1/2}$ as the induced norm. Then, given a sequence (x_k) in X, we can also define the strong convergence with respect to $\|\cdot\|$ and the weak convergence with respect to $\langle\cdot\,,\cdot\rangle$. These types of convergence are in general weaker than the previous ones, defined with respect to $\|\cdot, \ldots, \cdot\|$ and $\langle\cdot,\cdot \mid \cdot\,, \ldots, \cdot\rangle$ respectively.

In the standard case, however they are as strong as the previous ones, respectively, so that we have the following relation between the four types of convergence:

Strong convergence w.r.t $\|\cdot, \ldots, \cdot\|$ $\Rightarrow$ weal convergence w.r.t $\langle\cdot,\cdot \mid \cdot\,, \ldots, \cdot\rangle$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

Strong convergence w.r.t $\|\cdot\|$ $\Rightarrow$ weak convergence w.r.t $\langle\cdot\,,\cdot\rangle$

This gives us another explanation why our counter – example in the previous section works.

Finally, in the finite – dimensional case, any sequence that converges weakly with respect to $\langle\cdot\,,\cdot\rangle$ will converge strongly with respect to $\|\cdot\|$, and that any sequence that converges strongly with respect to $\|\cdot\|$ will converge strongly with respect to $\|\cdot, \ldots, \cdot\|$. Therefore, the four types of convergence are all equivalent.

Conclusion

In this paper, we have explored the concepts of strong and weak convergence within the framework of nnn-inner product spaces, a generalization that introduces additional complexity to the traditional notions of convergence. Through rigorous analysis, we have established that strong convergence in an nnn-inner product space implies weak convergence, aligning with classical results in inner product spaces. However, we demonstrated through counter-examples that the converse does not hold, highlighting the nuanced differences between these types of convergence in higher-dimensional settings.

The application of an analogue of Parseval's identity within nnn-inner product spaces provided a powerful tool to further illustrate these distinctions. By leveraging properties such as the triangle inequality, the Cauchy-Schwarz inequality, and continuity, we have shown the intricate relationships that govern convergence in these spaces. Additionally, we proved that a sequence cannot converge weakly to two distinct points, reinforcing the uniqueness of weak limits.

Our findings underscore the importance of understanding the specific conditions under which different types of convergence occur, particularly in the context of nnn-inner product spaces,

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where the interplay between geometry and analysis is more complex. These results have potential implications for further studies in functional analysis, quantum mechanics, and other fields that utilize advanced mathematical structures to model higher-order interactions. The exploration of special cases and the relations between different types of convergence also open avenues for future research, particularly in extending these concepts to other generalized spaces and their applications.

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